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Governing equations and general solutions of plane elasticity of one-dimensional quasicrystals

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Abstract

Plane elasticity theory of one-dimensional quasicrystals dealing with all point groups is investigated systematically. The governing equations of elastic fields and their general solutions are derived by the complex variable functions method. As an example, the elastic fields of a straight dislocation along the quasiperiodic axis of an orthorhombic quasicystal are calculated. The relevant singularity of the stress field for the dislocation in the quasicrystals is also discussed.

Keywords: One-dimensional quasicrystals; Plane elasticity; Governing equation; General solution; Dislocation

1. Introduction

Many experiments and theoretical analyses have shown that quasicrystals(QCs) are new materials with a complex structure and unusual properties (Ronchetti, 1987; Socolar et al., 1986; Wang et al., 1997; Fan, 1999; Fan and Mai, 2003 etc.). The discovery of this new solid structure and the production of large single grained QCs in various alloy systems with thermodynamical stability brings about not only a profound revolution in traditional theory of crystals, but also a challenge to the mathematical methods on describing and analyzing the structure quantitatively. Among various QCs, one-dimensional (1D) QCs are of particular interest for the researchers after the success of Merlin et al. (1985) in growing model systems, where quasi-periodicity is built up. From experimental side, it has been possible to construct Fibonacci superlattices by epitaxial growths methods (Merlin et al., 1985) and some stable 1D QCs have been obtained (Yang et al., 1996). From theoretical side, Wang et al. (1997) derived all the possible point groups and space groups of 1D QCs; Liu et al. (1997) studied the physical properties of 1D QCs. However, comparatively less

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works have been done on the theory of elasticity of 1D QCs. The difficulty to tackle these types of problems comes from more elastic constants as well as coupling phonon–phason fields. Although there are some research papers on elasticity theory of 1D QCs (Fan et al., 1999; Li and Fan, 1999; Peng and Fan, 2000; Peng et al., 2000; Peng and Fan, 2001; Liu et al., 2003), they involve only the elasticity theory of 1D hexagonal QCs with point group 6mm, which is the simplest class of 1D QCs. The present paper is devoted to general solutions of plane elasticity problems of 1D QCs, dealing with all point groups and its application.

Defectiveness of quasi-crystalline materials was observed (Zhang and Urban, 1989). It is well known that defects influence physical and mechanical properties of solid materials greatly. Experiments showed that QCs are quite brittle (Meng et al., 1994), and the brittle materials are sensitive to the defects. As an application of elastic theory of the QCs, one typical example of dislocation is investigated and the exact analytic solutions of the elastic fields are given.

2. Basic theory

A 1D QCs is defined as a three dimensional body of which the atom arrangement is periodic in a plane and quasi-periodic in the third direction. From Wang et al. (1997), there are 31 possible point groups in 1D QCs, which are divided into ten Laue classes and six systems, namely, triclinic, monoclinic, orthorhombic, tetragonal, trigonal and hexagonal system. In the case of plane elasticity, the body of QCs must have at least a symmetric plane. On the other hand, there exists at least one symmetric plane in all 1D QCs systems except triclinic QCs, so the investigation of the plane elasticity of 1D QCs is meaningful and extensive. In this paper, we assume the unique quasi-periodic axis of 1D QCs is axis x_3 in a rectilinear coordinate system (x_1, x_2, x_3).

A theoretical description of the deformed state of QCs requires a combined consideration of interrelated phonon and phason fields. Owing to the existence of phason fields, the elasticity of QCs is more complex than that of the conventional crystals. In QCs, a phason displacement field v exists in addition to a phonon displacement field $u(u_1, u_2, u_3)$. They have mutual interaction.

Let ε_{11}, ε_{22}, ε_{33}, ε_{23}, ε_{31}, ε_{12}, w_{33}, w_{31}, w_{32} denote the phonon strains ε_{ij} and phason strains w_{3j}, and σ_{11}, σ_{22}, σ_{33}, σ_{23}, σ_{31}, σ_{12}, H_{33}, H_{31}, H_{32} denote the phonon stress σ_{ij} and phason stress H_{3j}, respectively. Then the generalized Hooke's laws of the elasticity problem of 1D QCs are

$$\sigma_{ij} = C_{ijkl}\varepsilon_{kl} + R_{ij3l}w_{3l} \quad (i, j, k, l = 1, 2, 3), \tag{1}$$

$$H_{3j} = R_{kl3j}\varepsilon_{kl} + K_{3j3l}w_{3l} \tag{2}$$

and the static equilibrium equations in the absence of body forces are

$$\partial_1\sigma_{11} + \partial_2\sigma_{12} + \partial_3\sigma_{13} = 0, \quad \partial_1\sigma_{21} + \partial_2\sigma_{22} + \partial_3\sigma_{23} = 0, \tag{3}$$

$$\partial_1\sigma_{31} + \partial_2\sigma_{32} + \partial_3\sigma_{33} = 0, \quad \partial_1 H_{31} + \partial_2 H_{32} + \partial_3 H_{33} = 0. \tag{4}$$

Besides, geometry equations are given by

$$\varepsilon_{ij} = \frac{1}{2}(\partial_j u_i + \partial_i u_j), \quad w_{3j} = \partial_j v, \quad i, j = 1, 2, 3. \tag{5}$$

Here we have used tensor notation and $\partial_j = \partial/\partial x_j$, the same hereafter.

Eqs. (1)–(5) are the basic relations of elasticity theory of 1D QCs.

3. Plane elasticity and governing equation of monoclinic QCs

For monoclinic QCs, there are 25 elastic constants in all, namely, C_{1111}, C_{2222}, C_{3333}, C_{1122}, C_{1133}, C_{1112}, C_{2233}, C_{2212}, C_{3312}, C_{3232}, C_{3231}, C_{3131}, C_{1212} for phonon fields, K_{3333}, K_{3131}, K_{3232}, K_{3132} for phason fields, and R_{1133}, R_{2233}, R_{3333}, R_{1233}, R_{2331}, R_{2332}, R_{3131}, R_{3132} for coupling phonon–phason fields.

So the generalized Hooke's laws of monoclinic QCs are given by

$$
\begin{aligned}
\sigma_{11} &= C_{11}\varepsilon_{11} + C_{12}\varepsilon_{22} + C_{13}\varepsilon_{33} + 2C_{16}\varepsilon_{12} + R_1 w_{33},\\
\sigma_{22} &= C_{12}\varepsilon_{11} + C_{22}\varepsilon_{22} + C_{23}\varepsilon_{33} + 2C_{26}\varepsilon_{12} + R_2 w_{33},\\
\sigma_{33} &= C_{13}\varepsilon_{11} + C_{23}\varepsilon_{22} + C_{33}\varepsilon_{33} + 2C_{36}\varepsilon_{12} + R_3 w_{33},\\
\sigma_{23} &= \sigma_{32} = 2C_{44}\varepsilon_{23} + 2C_{45}\varepsilon_{31} + R_4 w_1 + R_5 w_{32},\\
\sigma_{31} &= \sigma_{13} = 2C_{45}\varepsilon_{23} + 2C_{55}\varepsilon_{31} + R_6 w_1 + R_7 w_{32},\\
\sigma_{12} &= \sigma_{21} = C_{16}\varepsilon_{11} + C_{26}\varepsilon_{22} + C_{36}\varepsilon_{33} + 2C_{66}\varepsilon_{12} + R_8 w_{33},\\
H_{31} &= 2R_4\varepsilon_{23} + 2R_6\varepsilon_{31} + K_1 w_{31} + K_4 w_{32},\\
H_{32} &= 2R_5\varepsilon_{23} + 2R_7\varepsilon_{31} + K_4 w_{31} + K_2 w_{32},\\
H_{33} &= R_1\varepsilon_{11} + R_2\varepsilon_{22} + R_3\varepsilon_{33} + 2R_8\varepsilon_{12} + K_3 w_{33}.
\end{aligned}
\tag{6}
$$

Here and subsequently we write the elastic constant C_{ijkl} in a contracted matrix notation C_{pq} as was done in the case of conventional crystal, and we have $K_{3131} = K_1$, $K_{3232} = K_2$, $K_{3333} = K_3$, $K_{3132} = K_4$, $R_{1133} = R_1$, $R_{2233} = R_2$, $R_{3333} = R_3$, $R_{2331} = R_4$, $R_{2332} = R_5$, $R_{3131} = R_6$, $R_{3132} = R_7$ and $R_{1233} = R_8$.

When the direction of defects such as infinitely long straight dislocations and cracks etc. is parallel to the quasi-periodic axis of 1D QCs, the geometry properties of the materials will not change along the quasi-periodic direction. If we take quasi-periodic axis of 1D QCs for axis x_3, then

$$\frac{\partial(\ )}{\partial x_3} = 0, \tag{7}$$

i.e. all fields variables depend only on coordinates x_1 and x_2. This is so-called plane elasticity (Fan, 1999).

Substitution of Eq. (7) in Eqs. (3)–(6) leads to two separate problems as follows:

Problem I

$$\sigma_{23} = \sigma_{32} = 2C_{44}\varepsilon_{23} + 2C_{45}\varepsilon_{31} + R_4 w_{31} + R_5 w_{32}, \tag{8}$$

$$\sigma_{31} = \sigma_{13} = 2C_{45}\varepsilon_{23} + 2C_{55}\varepsilon_{31} + R_6 w_{31} + R_7 w_{32}, \tag{9}$$

$$H_{31} = 2R_4\varepsilon_{23} + 2R_6\varepsilon_{31} + K_1 w_{31} + K_4 w_{32}, \tag{10}$$

$$H_{32} = 2R_5\varepsilon_{23} + 2R_7\varepsilon_{31} + K_4 w_{31} + K_2 w_{32}, \tag{11}$$

$$\varepsilon_{3j} = \varepsilon_{j3} = \frac{1}{2}\partial_j u_3, \quad w_{3j} = \partial_j v, \quad j = 1, 2, \tag{12}$$

$$\partial_1\sigma_{31} + \partial_2\sigma_{32} = 0, \quad \partial_1 H_{31} + \partial_2 H_{32} = 0. \tag{13}$$

This is a anti-plane elasticity problem for coupling phonon–phason fields.

Problem II

$$\sigma_{11} = C_{11}\varepsilon_{11} + C_{12}\varepsilon_{22} + 2C_{16}\varepsilon_{12}, \tag{14}$$

$$\sigma_{22} = C_{12}\varepsilon_{11} + C_{22}\varepsilon_{22} + 2C_{26}\varepsilon_{12}, \tag{15}$$

$$\sigma_{12} = \sigma_{21} = C_{16}\varepsilon_{11} + C_{26}\varepsilon_{22} + 2C_{66}\varepsilon_{12}, \tag{16}$$

$$\sigma_{33} = C_{13}\varepsilon_{11} + C_{23}\varepsilon_{22} + 2C_{36}\varepsilon_{12}, \tag{17}$$

$$H_{33} = R_1\varepsilon_{11} + R_2\varepsilon_{22} + 2R_8\varepsilon_{12}, \tag{18}$$

$$\varepsilon_{ij} = \frac{1}{2}(\partial_j u_i + \partial_i u_j) \quad i,j = 1,2, \tag{19}$$

$$\partial_1\sigma_{11} + \partial_2\sigma_{12} = 0, \quad \partial_1\sigma_{21} + \partial_2\sigma_{22} = 0. \tag{20}$$

This is similar to a plane strain problem for monoclinic crystal, but has an extra Eq. (18) here.

For Problem I, substituting Eq. (12) into Eqs. (8)–(11), then into Eq. (13), yields the equilibrium equations in terms of displacements as follows:

$$(C_{55}\partial_1^2 + C_{44}\partial_2^2 + 2C_{45}\partial_1\partial_2)u_3 + [R_6\partial_1^2 + R_5\partial_2^2 + (R_4 + R_7)\partial_1\partial_2]v = 0, \tag{21}$$

$$[R_6\partial_1^2 + R_5\partial_2^2 + (R_4 + R_7)\partial_1\partial_2]u_3 + (K_1\partial_1^2 + K_2\partial_2^2 + 2K_4\partial_1\partial_2)v = 0. \tag{22}$$

This is a set of partial differential equations for coupling phonon–phason fields, which is very different from that of conventional crystals. It seems to be extremely difficult to find the solution by means of direct integration due to the complexity of the equations. Now we introduce a displacement potential function to simplify above equations. Let

$$u_3 = [R_6\partial_1^2 + R_5\partial_2^2 + (R_4 + R_7)\partial_1\partial_2]F, \qquad v = -(C_{55}\partial_1^2 + C_{44}\partial_2^2 + 2C_{45}\partial_1\partial_2)F, \tag{23}$$

where $F(x_1, x_2)$ is the displacement potential function introduced. It is clear that Eq. (21) is satisfied.

Substituting Eq. (23) into Eq. (22), we have

$$(a_1\partial_1^4 + a_2\partial_1^3\partial_2 + a_3\partial_1^2\partial_2^2 + a_4\partial_1\partial_2^3 + a_5\partial_2^4)F = 0, \tag{24}$$

with constants

$$\begin{aligned} &a_1 = R_6^2 - K_1C_{55}, \quad a_2 = 2[R_6(R_4 + R_7) - K_1C_{45} - K_4C_{55}], \\ &a_3 = 2R_5R_6 + (R_4 + R_7)^2 - K_1C_{44} - K_2C_{55} - 4K_4C_{45}, \\ &a_4 = 2[R_5(R_4 + R_7) - K_2C_{45} - K_4C_{44}], \quad a_5 = R_5^2 - K_2C_{44}. \end{aligned} \tag{25}$$

Eq. (24) indicates that the terminal governing equation of Problem I is a fourth-order partial differential equation. Furthermore, substitution of Eq. (23) in Eqs. (8)–(11) yields

$$\sigma_{32} = [(R_6C_{45} - R_4C_{55})\partial_1^3 + (R_6C_{44} - R_4C_{45} + R_7C_{45} - R_5C_{55})\partial_1^2\partial_2 + (R_7C_{44} - R_5C_{45})\partial_1\partial_2^2]F, \tag{26}$$

$$\sigma_{31} = [(R_5C_{45} - R_7C_{44})\partial_2^3 + (R_5C_{55} - R_7C_{45} + R_4C_{45} - R_6C_{44})\partial_2^2\partial_1 + (R_4C_{55} - R_6C_{45})\partial_2\partial_1^2]F, \tag{27}$$

$$H_{32} = [R_6\partial_1^2 + R_5\partial_2^2 + (R_4 + R_7)\partial_1\partial_2](R_5\partial_2 + R_7\partial_1)F - (C_{55}\partial_1^2 + C_{44}\partial_2^2 + 2C_{45}\partial_1\partial_2)(K_4\partial_1 + K_2\partial_2)F, \tag{28}$$

$$H_{31} = [R_6\partial_1^2 + R_5\partial_2^2 + (R_4 + R_7)\partial_1\partial_2](R_4\partial_2 + R_6\partial_1)F - (C_{55}\partial_1^2 + C_{44}\partial_2^2 + 2C_{45}\partial_1\partial_2)(K_1\partial_1 + K_4\partial_2)F. \tag{29}$$

These are the representations of stress components in terms of the displacement potential function corresponding to Problem I.

For Problem II, substituting Eq. (19) into Eqs. (14)–(16), then substituting the obtained results into Eq. (20), yields the equilibrium equations in terms of displacements as following:

$$(C_{11}\partial_1^2 + C_{66}\partial_2^2 + 2C_{16}\partial_1\partial_2)u_1 + [C_{16}\partial_1^2 + C_{26}\partial_2^2 + (C_{12} + C_{66})\partial_1\partial_2]u_2 = 0, \tag{30}$$

$$[C_{16}\partial_1^2 + C_{26}\partial_2^2 + (C_{12} + C_{66})\partial_1\partial_2]u_1 + (C_{66}\partial_1^2 + C_{22}\partial_2^2 + 2C_{26}\partial_1\partial_2)u_2 = 0. \tag{31}$$

Like Problem I, we introduce a displacement potential function to simplify them. Let

$$u_1 = [C_{16}\partial_1^2 + C_{26}\partial_2^2 + (C_{12} + C_{66})\partial_1\partial_2]G, \quad u_2 = -(C_{11}\partial_1^2 + C_{66}\partial_2^2 + 2C_{16}\partial_1\partial_2)G, \tag{32}$$

where $G(x_1, x_2)$ is another displacement potential function introduced. It is clear that Eq. (30) is satisfied. Substituting Eq. (32) into Eq. (31), we have

$$(c_1\partial_1^4 + c_2\partial_1^3\partial_2 + c_3\partial_1^2\partial_2^2 + c_4\partial_1\partial_2^3 + c_5\partial_2^4)G = 0, \tag{33}$$

with constants

$$\begin{aligned} &c_1 = C_{16}^2 - C_{11}C_{66}, \quad c_2 = 2(C_{16}C_{12} - C_{11}C_{26}), \\ &c_3 = C_{12}^2 - 2C_{16}C_{26} + 2C_{12}C_{66} - C_{11}C_{22}, \\ &c_4 = 2(C_{26}C_{12} - C_{16}C_{22}), \quad c_5 = C_{26}^2 - C_{22}C_{66}. \end{aligned} \tag{34}$$

Eq. (33) indicates that the terminal governing equations of Problem II is also a fourth-order partial differential equation. Furthermore, substitution of Eq. (32) in Eqs. (14)–(16) yields

$$\sigma_{11} = [(C_{11}C_{66} - C_{16}^2)\partial_1^2\partial_2 + (C_{11}C_{26} - C_{16}C_{12})\partial_1\partial_2^2 + (C_{16}C_{26} - C_{12}C_{66})\partial_2^3]G, \tag{35}$$

$$\sigma_{22} = [(C_{12}C_{16} - C_{11}C_{26})\partial_1^3 + (C_{12}^2 + C_{12}C_{66} - C_{16}C_{26} - C_{11}C_{22})\partial_1^2\partial_2 - 2C_{16}C_{26}\partial_1^2\partial_2 + (C_{26}^2 - C_{22}C_{66})\partial_2^3]G, \tag{36}$$

$$\sigma_{12} = [(C_{16}^2 - C_{11}C_{26})\partial_1^3 + (C_{12}C_{16} - C_{11}C_{26})\partial_1^2\partial_2 + (C_{12}C_{66} - C_{16}C_{26})\partial_1\partial_2^2]G. \tag{37}$$

Similarly, σ_{33} and H_{33} can also be obtained, which are omitted here because we consider the stresses in the x_1–x_2-plane only.

These are the representations of stress components in terms of the displacement potential function corresponding to Problem II.

Hence, the plane elasticity of 1D monoclinic QCs is governed by two fourth-order partial differential equations (24) and (33).

4. General solution for monoclinic QCs

Following Sosa (1991) and Lekhnitskii (1963), the solution of the fourth-order partial differential equation (24) can be represented by two analytic functions $F_k(z_k)(k = 1, 2)$ as following

$$F(x_1,x_2)=2\mathrm{Re}\sum_{k=1}^{2}F_k(z_k),\quad z_k=x_1+\mu_k x_2, \tag{38}$$

where Re denotes the real part of corresponding complex expression; $\mu_k=\alpha_k+\mathrm{i}\beta_k$ (with $\mathrm{i}=\sqrt{-1}$, $k=1,2$) are distinct complex parameters to be determined by the characteristic equation

$$a_5\mu^4+a_4\mu^3+a_3\mu^2+a_2\mu+a_1=0, \tag{39}$$

and $\mu_1\neq\mu_2$.

If the roots of Eq. (39) are multi-roots, namely $\mu_1=\mu_2$, we have

$$F(x_1,x_2)=2\mathrm{Re}[F_1(z_1)+\overline{z_1}F_2(z_1)],\quad z_1=x_1+\mu_1x_2. \tag{40}$$

The $\mu_k(k=1,2)$ can be, in principle, calculated analytically when the elastic constants of the QCs are given.

Substitution of Eq. (38) in Eqs. (23) and (26)–(29) yields

$$u_3=2\mathrm{Re}\sum_{k=1}^{2}[R_6+(R_4+R_7)\mu_k+R_5\mu_k^2]f_k(z_k), \tag{41}$$

$$v=-2\mathrm{Re}\sum_{k=1}^{2}(C_{55}+2C_{45}\mu_k+C_{44}\mu_k^2)f_k(z_k), \tag{42}$$

$$\sigma_{32}=2\mathrm{Re}\sum_{k=1}^{2}[R_6C_{45}-R_4C_{55}+(R_6C_{44}-R_4C_{45}+R_7C_{45}-R_5C_{55})\mu_k+(R_7C_{44}-R_5C_{45})\mu_k^2]f_k'(z_k), \tag{43}$$

$$\sigma_{31}=2\mathrm{Re}\sum_{k=1}^{2}[R_4C_{55}-R_6C_{45}+(R_5C_{55}+R_4C_{45}-R_7C_{45}-R_6C_{44})\mu_k+(R_5C_{45}-R_7C_{44})\mu_k^2]\mu_kf_k'(z_k), \tag{44}$$

$$H_{32}=2\mathrm{Re}\sum_{k=1}^{2}[(R_7+R_5\mu_k)(R_6+R_4\mu_k+R_7\mu_k+R_5\mu_k^2)-(K_4+K_2\mu_k)(C_{55}+2C_{45}\mu_k+C_{44}\mu_k^2)]f_k'(z_k), \tag{45}$$

$$H_{31}=2\mathrm{Re}\sum_{k=1}^{2}[(R_6+R_4\mu_k)(R_6+R_4\mu_k+R_7\mu_k+R_5\mu_k^2)-(K_1+K_4\mu_k)(C_{55}+2C_{45}\mu_k+C_{44}\mu_k^2)]f_k'(z_k), \tag{46}$$

where $f_k(z_k)=\partial_{z_k}^2F_k(z_k)=F_k''(z_k)$.

Similarly,

$$G(x_1,x_2)=2\mathrm{Re}\sum_{k=1}^{2}G_k(\xi_k),\quad \xi_k=x_1+\lambda_kx_2, \tag{47}$$

where $\xi_k=\alpha_{k1}+\mathrm{i}\beta_{k1}$ (with $\mathrm{i}=\sqrt{-1},k=1,2$) are also distinct complex parameters to be determined by the characteristic equation

$$c_5\lambda^4+c_4\lambda^3+c_3\lambda^2+c_2\lambda+c_1=0. \tag{48}$$

And

$$u_1 = 2\mathrm{Re}\sum_{k=1}^{2}[C_{16} + (C_{12} + C_{66})\lambda_k + C_{26}\lambda_k^2]g_k(\xi_k), \tag{49}$$

$$u_2 = -2\mathrm{Re}\sum_{k=1}^{2}(C_{11} + 2C_{16}\lambda_k + C_{66}\lambda_k^2)g_k(\xi_k), \tag{50}$$

$$\sigma_{11} = 2\mathrm{Re}\sum_{k=1}^{2}[C_{66}C_{11} - C_{16}^2 + (C_{11}C_{26} - C_{12}C_{16})\lambda_k + (C_{16}C_{26} - C_{12}C_{66})\lambda_k^2]\lambda_k g_k'(\xi_k), \tag{51}$$

$$\sigma_{22} = 2\mathrm{Re}\sum_{k=1}^{2}[C_{16}C_{12} - C_{11}C_{26} + (C_{26}^2 + C_{12}C_{66} - C_{16}C_{26} - C_{11}C_{22})\lambda_k - 2C_{16}C_{22}\lambda_k^2 + (C_{26}^2 - C_{22}C_{66})\lambda_k^3]g_k'(\xi_k), \tag{52}$$

$$\sigma_{12} = 2\mathrm{Re}\sum_{k=1}^{2}[C_{16}^2 - C_{11}C_{66} + (C_{16}C_{12} - C_{11}C_{26})\lambda_k + (C_{12}C_{66} - C_{16}C_{26})\lambda_k^2]g_k'(\xi_k), \tag{53}$$

where $g_k(\xi_k) = \partial_{\xi_k}^2 G_k(\xi_k) = G_k''(\xi_k)$.

These are the complex variable representations of displacement and stress components of phonon fields and phason fields in 1D monoclinic QCs. They are so-called general solutions of plane elasticity of 1D monoclinic QCs. With this general solutions, it is very convenient to give the special solution of some dislocation and crack problems of 1D QCs.

5. Other 1D QC systems

5.1. Orthorhombic QC system

In the sequel, the meaning of symbol for the point groups is the same as in (Wang et al., 1997). Orthorhombic QCs comprise the point groups 2_h2_h2, mm2, $2_h\mathrm{mm}_h$ and mmm_h, which belong to one Laue class. Owing to the increase of symmetric elements of orthorhombic QCs in comparison with monoclinic QCs, one has also

$$C_{16} = C_{26} = C_{36} = C_{45} = K_4 = R_4 = R_7 = R_8 = 0. \tag{54}$$

Therefore the number of non-zero elastic constants of orthorhombic QCs reduces to 17, namely, C_{11}, C_{22}, C_{33}, C_{12}, C_{13}, C_{23}, C_{44}, C_{55}, C_{66} for phonon fields; K_1, K_2, K_3 for phason fields; R_1, R_2, R_3, R_5, R_6 for coupling phonon–phason fields.

Substituting (54) into Eqs. (25) and (34), we find that a_1 and a_5 are the same as in (25), and

$$\begin{aligned} &a_2 = a_4 = 0, \quad a_3 = 2R_5R_6 - K_1C_{44} - K_2C_{55}, \\ &c_1 = -C_{11}C_{66}, \quad c_2 = c_4 = 0, \quad c_5 = -C_{22}C_{66}, \\ &c_3 = C_{12}^2 + 2C_{12}C_{66} - C_{11}C_{22}. \end{aligned} \tag{55}$$

Furthermore, all fields variables are simplified as follows

$$u_1 = 2(C_{12} + C_{66})\mathrm{Re}\sum_{k=1}^{2}\lambda_k g_k(\xi_k), \qquad u_2 = -2\mathrm{Re}\sum_{k=1}^{2}(C_{11} + C_{66}\lambda_k^2)g_k(\xi_k), \tag{56}$$

$$u_3 = 2\mathrm{Re}\sum_{k=1}^{2}(R_6 + R_5\mu_k^2)f_k(z_k), \qquad v = -2\mathrm{Re}\sum_{k=1}^{2}(C_{55} + C_{44}\mu_k^2)f_k(z_k), \tag{57}$$

$$\sigma_{11} = 2\mathrm{Re}\sum_{k=1}^{2}(C_{66}C_{11} - C_{12}C_{66}\lambda_k^2)\lambda_k g_k'(\xi_k), \tag{58}$$

$$\sigma_{22} = 2\mathrm{Re}\sum_{k=1}^{2}[(C_{12}^2 + C_{12}C_{66} - C_{11}C_{22}) - C_{22}C_{66}\lambda_k^2]\lambda_k g_k'(\xi_k), \tag{59}$$

$$\sigma_{12} = 2\mathrm{Re}\sum_{k=1}^{2}(-C_{11}C_{66} + C_{12}C_{66}\lambda_k^2)g_k'(\xi_k), \tag{60}$$

$$\sigma_{32} = 2\mathrm{Re}\sum_{k=1}^{2}(R_6C_{44} - R_5C_{55})\mu_k f_k'(z_k), \tag{61}$$

$$\sigma_{31} = 2(R_5C_{55} - R_6C_{44})\mathrm{Re}\sum_{k=1}^{2}\mu_k^2 f_k'(z_k), \tag{62}$$

$$H_2 = 2\mathrm{Re}\sum_{k=1}^{2}[R_5R_6 - K_2C_{55} + (R_5^2 - K_2C_{44})\mu_k^2]\mu_k f_k'(z_k), \tag{63}$$

$$H_1 = 2\mathrm{Re}\sum_{k=1}^{2}[R_6^2 - K_1C_{55} + (R_5R_6 - K_1C_{44})\mu_k^2]f_k'(z_k). \tag{64}$$

5.2. Tetragonal QC system

1D tetragonal QCs divide into two Laue classes. Point groups $\bar{4}2_\mathrm{h}\mathrm{m}$, 4mm, $42_\mathrm{h}2_\mathrm{h}$ and $4/\mathrm{m}_\mathrm{h}\mathrm{mm}$ in the QCs belong to one Laue class. Owing to the increase of new symmetric elements of tetragonal QCs, besides (54), one has also

$$C_{11} = C_{22}, \quad C_{13} = C_{23}, \quad C_{44} = C_{55}, \quad K_1 = K_2, \quad R_1 = R_2, \quad R_5 = R_6. \tag{65}$$

Therefore the number of non-zero elastic constants of 1D tetragonal QCs reduces to 11.

Substituting (65) into Eqs. (55)–(64), simplified forms of the governing equations and general solutions will be obtained.

Point groups $\bar{4}$, 4 and $4/\mathrm{m}_\mathrm{h}$ in tetragonal QC system belong to another Laue class, of which the plane elasticity can be simplified with the same process.

5.3. Hexagonal QC system

1D hexagonal QCs divide into two Laue classes. The point groups 6, $\bar{6}$ and 6/m$_h$ mmm$_h$ belong to one, and 62$_h$2$_h$, 6mm, $\bar{6}$m2$_h$ and 6/m$_h$mm belong to another. Owing to the increase of symmetric elements of hexagonal QCs in comparison with monoclinic QCs, for the point groups 6, $\bar{6}$ and 6/m$_h$ mmm$_h$, we have also

$$\begin{aligned} &C_{11} = C_{22}, \quad C_{13} = C_{23}, \quad C_{44} = C_{55}, \quad 2C_{66} = C_{11} - C_{12}, \quad C_{16} = C_{26} = C_{36} = C_{45} = 0; \\ &K_1 = K_2, \quad K_4 = 0; \qquad R_1 = R_2, \quad R_5 = R_6, \quad R_4 = -R_7, \quad R_8 = 0. \end{aligned} \tag{66}$$

Therefore the number of their non-zero elastic constants reduces to 11, namely C_{11}, C_{33}, C_{12}, C_{13}, C_{44} for phonon fields; K_3, K_1 for phason fields; R_1, R_3, R_4, R_6 for coupling phonon–phason fields. Note that although the number of two classes of QCs in Sections 5.2 and 5.3 is same, the components of them are different, consequently the governing equation and general solution are also different.

For the point groups 62$_h$2$_h$, 6mm, $\bar{6}$m2$_h$ and 6/m$_h$mm, besides (66), we have also $R_4 = 0$.

Substituting (66) into Eqs. (25) and (34), we get

$$a_1 = a_5 = R_5^2 - K_1C_{44}, \quad a_2 = a_4 = 0, \quad a_3 = 2(R_5^2 - K_1C_{44}); \tag{67}$$

$$c_1 = c_5 = -C_{11}C_{66}, \quad c_2 = c_4 = 0, \quad c_3 = -2C_{11}C_{66}. \tag{68}$$

Substitution of (67) in Eq. (24) and (68) in Eq. (33), yields the governing equations as follows

$$\nabla^2\nabla^2 F = 0; \quad \nabla^2\nabla^2 G = 0, \tag{69}$$

where $\nabla^2 = \partial_1^2 + \partial_2^2$.

Eq. (69) is the well-known bi-harmonic equation in classical elasticity. In this case, the characteristic equation of (69) is $\mu^4 + 2\mu^2 + 1 = 0$. So $\mu = \pm i$, and $F = 2\mathrm{Re}[F_1(z) + \bar{z}F_2(z)]$, $G = 2\mathrm{Re}[G_1(z) + \bar{z}G_2(z)]$. $F_i(z)$ and $G_i(z)$ are the well-known complex potential functions (Muskhelishvili, 1963). Furthermore, there are many methods such as complex potential method, Riemann–Hilbert boundary value method and Fourier transform method etc. to solve the bi-harmonic equation in classical elastic theory. Specially, under the condition of $C_{44}K_1 - R_5 \neq 0$, Eq. (24) gives the result obtained by Fan (1999).

6. An example

As an application of above theory, we investigate a typical example of dislocation in the orthorhombic QCs. The dislocation of QCs is described by Burgers vector in higher dimensional space (Ding et al., 1998; Bohsung and Trebin, 1989). The dislocation of 1D QCs can be expressed by Burgers vector in four–dimensional space. Consider an infinitely long straight dislocation parallel to the quasi-periodic direction in an infinite body of 1D orthorhombic QCs. Then, due to the symmetry of the QCs, the problem belongs to plane elasticity. Assume a dislocation located at point z_0 in the x_1–x_2-plane and its Burgers vector is $(b_1, b_2, b_3, b_\perp)$. By means of the superposition principle, we have

$$(b_1, b_2, b_3, b_\perp) = (b_1, 0, 0, 0) + (0, b_2, 0, 0) + (0, 0, b_3, 0) + (0, 0, 0, b_\perp).$$

From (Ding et al., 1998; Bohsung and Trebin, 1989), the dislocation conditions are given by

$$\oint_L \mathrm{d}u_1 = b_1, \quad \oint_L \mathrm{d}u_2 = b_2, \quad \oint_L \mathrm{d}u_3 = b_3, \quad \oint_L \mathrm{d}v = b_\perp, \tag{70}$$

where L denotes a Burgers contour surrounding the dislocation z_0. Note that all four integrals are made in physical space.

We consider the case of vanishing boundary conditions in infinity only. Substituting Eqs. (56) and (57) into Eq. (70), through some derivation, yields

$$f_1'(z_1) = -\frac{[(C_{55}+C_{44}\mu_2^2)b_3 + (R_6+R_5\mu_2^2)b^{\perp}]\mathrm{i}}{4\pi(\mu_1^2-\mu_2^2)(R_5C_{55}-R_6C_{44})}\frac{1}{z_1-z_0}, \tag{71}$$

$$f_2'(z_2) = \frac{[(C_{55}+C_{44}\mu_1^2)b_3 + (R_6+R_5\mu_1^2)b^{\perp}]\mathrm{i}}{4\pi(\mu_1^2-\mu_2^2)(R_5C_{55}-R_6C_{44})}\frac{1}{z_2-z_0}, \tag{72}$$

$$g_1'(\xi_1) = -\frac{[(C_{11}+C_{66}\lambda_2^2)b_1 + (C_{12}+C_{66})\lambda_2 b_2)]\mathrm{i}}{4\pi(\lambda_1-\lambda_2)(C_{11}-C_{66}\lambda_1\lambda_2)(C_{12}+C_{66})}\frac{1}{\xi_1-z_0}, \tag{73}$$

$$g_2'(\xi_2) = \frac{[(C_{11}+C_{66}\lambda_1^2)b_1 + (C_{12}+C_{66})\lambda_1 b_2]\mathrm{i}}{4\pi(\lambda_1-\lambda_2)(C_{11}-C_{66}\lambda_1\lambda_2)(C_{12}+C_{66})}\frac{1}{\xi_2-z_0}. \tag{74}$$

Integrating Eqs. (71)–(74), then substituting them into Eqs. (56) and (57), yields the displacement components of phonon and phason fields for a dislocation in 1D orthorhombic QCs. By direct substitution of Eqs. (71)–(74) in Eqs. (58)–(64), we will obtain all stress components of the elastic fields. So the problem of dislocation of orthorhombic QCs is solved.

7. Conclusion and discussion

Now we summarize the main contents of this paper. First, plane elasticity theory of 1D QCs dealing with various point groups is established, and the governing equations in terms of the displacement potential functions are given. Second, general solutions expressed by complex variable functions are presented, which provide the foundation of solving some boundary value problems of QCs. Third, as an application of the general solutions, the dislocation problem in 1D orthorhombic QCs is investigated and its exact analytic solutions are obtained. The result indicates that a straight dislocation in 1D QCs includes two parts: one corresponds to purely edge type, and another corresponds to screw type of coupling phonon–phason fields. The two parts exist independently, and are superimposed. Meanwhile, the stresses for a straight dislocation still own one order singularity as in conventional elasticity, but are related also with the Burgers vector of phason fields. This property shows that the dislocation of QCs results from the mistake of its atom arrangement, because the phason stands for the atom arrangement of the QCs. Finally, as an open problem, we don't know if the case of the roots μ_k being real number is true for QCs. This is not the case for ordinary elastic material (Sosa, 1991).

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